The analysis of transient scattering for rectangular incident waves using the discrete Laguerre transforms

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Abstract: By expressing the transient electromagnetic behaviors in terms of continuous orthonormal Laguerre polynomials, we can obtain an unconditionally stable solution of the time domain electric field integral equation (TD-EFIE) for three-dimensional (3-D) arbitrary shaped conducting bodies. Besides using Gaussian type pulses, rectangular and triangular pulses are also used as incident waves in this method. However, because of the discontinuity in a rectangular pulse, Gibbs phenomenon will occur around the point of discontinuity when a continuous basis functions are used to approximate the incident wave in a least square sense. Noting that we deal with discrete data during our computation in a computer, we introduce the discrete Laugerre functions to solve TD-EFIE. They are exactly othonormal in a discrete sense. In this paper, we use the discrete Laguerre basis functions to approximate its continuous counterparts and then use them to express the rectangular incident wave and the response. Simulation results show that there's no Gibbs phenomenon. Furthermore, the computation of the Laguerre transform of the incident wave is more efficient.

Keywords: Laguerre functions, discrete Laugerre functions, EFIE, Gibbs phenomenon

1. TD-EFIE

Let S denote the surface of an electric conducting body illuminated by a transient electromagnetic wave. Since the total tangential electric field is zero on the conducting surface for all times, we have

$$\left[\frac{\partial}{\partial t}\mathbf{A}(\mathbf{r},t)+\nabla\Phi(\mathbf{r},t)\right]_{\tan}=\left[\mathbf{E}^{i}(\mathbf{r},t)\right]_{\tan},\ \mathbf{r}\in S \quad (1)$$

where $\mathbf{E}^i$ is the incident field and $\mathbf{A}$ and Φ are the magnetic vector potential and the electric scalar potential given by

$$\mathbf{A}(\mathbf{r},t)=\frac{\mu}{4\pi}\int_S\frac{\mathbf{J}(\mathbf{r}',\tau)}{R}dS' \quad (2)$$

$$\Phi(\mathbf{r},t)=\frac{1}{4\pi\varepsilon}\int_S\frac{q(\mathbf{r}',\tau)}{R}dS' \quad (3)$$

In (2) and (3), $R=\left|\mathbf{r}-\mathbf{r}'\right|$ represents the distance between the arbitrarily located observation point $\mathbf{r}$ and the source point $\mathbf{r}'$. $\tau=t-R/c$ is the retarded time. q is the surface charge density. μ and ε are the permeability

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and the permittivity of the medium, and c is the velocity of the propagation of the electromagnetic wave in that space. Equation (1) with (2) and (3) constitutes a TD-EFIE from which the unknown current $\mathbf{J}$ may be determined.

The surface of the structure to be analyzed is approximated by planar triangular patches. We define the spatial basis function associated with the n-*th* common edge as

$$\mathbf{f}_n(\mathbf{r}) = \mathbf{f}_n^+(\mathbf{r}) + \mathbf{f}_n^-(\mathbf{r}) \quad , \qquad \mathbf{f}_n^\pm(\mathbf{r}) = \begin{cases} \dfrac{l_n}{2A_n^\pm}\boldsymbol{\rho}_n^\pm\,, & \mathbf{r} \in T_n^\pm \\ 0\,, & \mathbf{r} \notin T_n^\pm \end{cases} \tag{4}$$

where l_n and $A_n^\pm$ are the length of the edge and the area of triangle $T_n^\pm$. $\boldsymbol{\rho}_n^\pm$ is the position vector defined with respect to the free vertex of $T_n^\pm$. The electric current $\mathbf{J}$ on the scattering structure may be approximated in terms of the vector basis function as

$$\mathbf{J}(\mathbf{r}, t) = \sum_{n=1}^{N} J_n(t)\mathbf{f}_n(\mathbf{r}) \tag{5}$$

where N represents the number of common edges, discounting the boundary edges in the triangulated model of the conducting object. We introduce a new source vector $\mathbf{e}(\mathbf{r}, t)$ defined by

$$\mathbf{J}(\mathbf{r}, t) = \frac{\partial}{\partial t}\mathbf{e}(\mathbf{r}, t)\,. \tag{6}$$

By using (5) and (6), we may express

$$\mathbf{e}(\mathbf{r}, t) = \sum_{n=1}^{N} e_n(t)\mathbf{f}_n(\mathbf{r})\,. \tag{7}$$

We now solve (1) by applying Galerkin's method in the MoM context and hence the testing functions are same as the expansion functions. By choosing the spatial expansion function $\mathbf{f}_m(\mathbf{r})$ also as the spatial testing functions, we have from (1)

$$< \mathbf{f}_m(\mathbf{r}), \frac{\partial}{\partial t}\mathbf{A}(\mathbf{r}, t) > + < \mathbf{f}_m(\mathbf{r}), \nabla\Phi(\mathbf{r}, t) > = < \mathbf{f}_m(\mathbf{r}), \mathbf{E}^{\mathrm{i}}(\mathbf{r}, t) > \tag{8}$$

where $m = 1, 2, \cdots, N$. In computing (8), we assume that the unknown transient quantity does not change appreciably within the triangle so that

$$\tau = t - \frac{R}{c} \quad \rightarrow \quad \tau_{mn}^{pq} = t - \frac{R_{mn}^{pq}}{c}\,,\; R_{mn}^{pq} = \left|\mathbf{r}_m^{cp} - \mathbf{r}_n^{cq}\right| \tag{9}$$

where p and q are + or -. $\mathbf{r}_m^{c\pm}$ is the position vector of the center in triangle $T_n^\pm$. With the assumption (9), (8) can be written as

$$\sum_{n=1}^{N}\sum_{p,q}\left[\mu a_{mn}^{pq}\frac{d^2}{dt^2}e_n(\tau_{mn}^{pq}) + \frac{b_{mn}^{pq}}{\varepsilon}e_n(\tau_{mn}^{pq})\right] = V_m^E(t) \tag{10}$$

where

$$a_{mn}^{pq} = \frac{1}{4\pi}\int_S \mathbf{f}_m^p(\mathbf{r}) \cdot \int_S \frac{\mathbf{f}_n^q(\mathbf{r}')}{R}dS'dS\,. \tag{11}$$

$$b_{mn}^{pq} = \frac{1}{4\pi}\int_S \nabla\cdot\mathbf{f}_m^p(\mathbf{r})\int_S \frac{\nabla'\cdot\mathbf{f}_n^q(\mathbf{r}')}{R}dS'dS\,. \tag{12}$$

$$V_m^E(t) = \int_S \mathbf{f}_m(\mathbf{r})\cdot\mathbf{E}^{\mathrm{i}}(\mathbf{r}, t)dS\,. \tag{13}$$

2. Continuous Laguerre Polynomials and the formulation

An orthonormal basis function set can be derived from the Laguerre functions through the representation ([2], [3])

$$\phi_j(st) = \sqrt{s}e^{-st/2}L_j(st) \tag{14}$$

where $L_j(t)$ is the Laguerre polynomial of degree j and s is the scaling factor. The continuous Laugerre transform for $e_n(t)$ can be given as

$$e_{n,j} = \int_0^\infty e_n(t)\phi_j(st)dt\,,\; e_n(t) = \sum_{j=0}^{\infty} e_{n,j}\phi_j(st) \tag{15}$$

The expressions of expanding the first and the second derivative of the transient coefficient are given as, respectively,

$$\frac{d}{dt}e_n(t) = s\sum_{j=0}^{\infty}\left[\frac{1}{2}e_{n,j} + \sum_{k=0}^{j-1} e_{n,k}\right]\phi_j(st) \tag{16}$$

$$\frac{d^2}{dt^2}e_n(t) = s^2\sum_{j=0}^{\infty}\left[\frac{1}{4}e_{n,j} + \sum_{k=0}^{j-1}(j-k)e_{n,k}\right]\phi_j(st)\,. \tag{17}$$

Substituting (17) and (15) into (10) and taking a temporal testing with $\phi_i(st)$, we have

$$\sum_{n=1}^{N}\sum_{p,q}\sum_{j=0}^{\infty}\left[\left(\frac{s^2}{4}\mu a_{mn}^{pq} + \frac{b_{mn}^{pq}}{\varepsilon}\right)e_{n,j} + s^2\mu a_{mn}^{pq}\sum_{k=0}^{j-1}(j-k)e_{n,k}\right]I_{ij}\left(s\frac{R_{mn}^{pq}}{c}\right) = V_{m,i}^{E} \tag{18}$$

where

$$I_{ij}\left(s\frac{R_{mn}^{pq}}{c}\right) = \int_0^\infty \phi_i(st)\phi_j\left(st - s\frac{R_{mn}^{pq}}{c}\right)d(st) \tag{19}$$

$$V_{m,i}^{E} = \int_0^\infty \phi_i(st)V_m^E(t)\,d(st)\,. \tag{20}$$

We note that $I_{ij} = 0$ when $j > i$. Therefore we can write the upper limit of the third summation symbol as i instead of ∞ in (18). In this result, moving the terms including $e_{n,j}$, which is known for $j < i$, to the right-hand side, we obtain

$$\begin{aligned}\sum_{n=1}^{N}\sum_{p,q}\left(\frac{s^2}{4}\mu a_{mn}^{pq} + \frac{b_{mn}^{pq}}{\varepsilon}\right)e_{n,i}I_{ii}\left(s\frac{R_{mn}^{pq}}{c}\right) = V_{m,i}^{E} &- \sum_{n=1}^{N}\sum_{p,q}\sum_{j=0}^{i-1}\left(\frac{s^2}{4}\mu a_{mn}^{pq} + \frac{b_{mn}^{pq}}{\varepsilon}\right)e_{n,j}I_{ij}\left(s\frac{R_{mn}^{pq}}{c}\right)\\ &- \sum_{n=1}^{N}\sum_{p,q}\sum_{j=0}^{i}s^2\mu a_{mn}^{pq}\sum_{k=0}^{j-1}(j-k)e_{n,k}I_{ij}\left(s\frac{R_{mn}^{pq}}{c}\right).\end{aligned} \tag{21}$$

Rewriting (21) in a simple form, we have

$$\sum_{n=1}^{N}\alpha_{mn}^{E}e_{n,i} = V_{m,i}^{E} + P_{m,i}^{E} \tag{22}$$

where

$$\alpha_{mn}^{E} = \sum_{p,q} \left(\frac{s^2}{4} \mu a_{mn}^{pq} + \frac{b_{mn}^{pq}}{\varepsilon} \right) \exp\left(-s \frac{R_{mn}^{pq}}{2c} \right) \tag{23}$$

$$P_{m,i}^{E} = -\sum_{n=1}^{N} \sum_{p,q} \left[\left(\frac{s^2}{4} \mu a_{mn}^{pq} + \frac{b_{mn}^{pq}}{\varepsilon} \right) \sum_{j=0}^{i-1} e_{n,j} I_{ij} \left(s \frac{R_{mn}^{pq}}{c} \right) + s^2 \mu a_{mn}^{pq} \sum_{j=0}^{i} \sum_{k=0}^{j-1} (j-k) e_{n,k} I_{ij} \left(s \frac{R_{mn}^{pq}}{c} \right) \right]. \tag{24}$$

It is important to note that $[\alpha_{mn}^{E}]$ is not a function of the degree of the temporal testing function. Therefore, we can obtain the unknown coefficients by solving (22) through an increase in the degree of the temporal testing functions.

3. The discrete Laugerre basis functions and the transforms for rectangular pulse

The discrete Laguerre functions, defined in the Z domain, can be written as ([1])

$$\Psi_j(z,a) = \sqrt{1-a^2}\, \frac{(a-z^{-1})^j}{(1-az^{-1})^{j+1}}; \;\; |a|<1, \qquad j = 0, 1, 2, \ldots \tag{25}$$

The constraint on the pole $|a|<1$ is set to make the functions causal and stable. It has the following recursive form in time domain

$$\psi_{j+1}(k) = a\psi_j(k) - \psi_j(k-1) + a\psi_{j+1}(k-1) \tag{26}$$

The discrete Laguerre functions are orthonormal in a discrete sense, which means

$$T_{ij} = \sum_{k=0}^{\infty} \psi_i(k,a)\psi_j(k,a) = \frac{1}{2\pi j} \oint_C \Psi_i(z,a)\Psi_j(1/z,a) \frac{dz}{z} = \delta_{ij} \tag{27}$$

The integration is along the unit circle in Z plane. In the limit Δt approaches zero, The sample value $\phi_j(sk \cdot \Delta t)$ and $\psi_j(k,a)$ become equal as they are related by

$$a = e^{-\frac{s\cdot\Delta t}{2}}, \text{ or } s = -\frac{2}{\Delta t} \ln a \tag{28}$$

where Δt is the sampling interval. The first 5 orders of continuous and discrete Laguerre functions for $a = 0.9$ and $\Delta t = 1$ are shown in Fig. 1, where the lines are the continuous functions and dots are the discrete ones.

The discrete Laguerre transform can be defined as

$$d_{n,j} = \sum_{k=0}^{\infty} e_n(k)\psi_j(k,a), \;\; e_n(k) = \sum_{j=0}^{\infty} d_{n,j}\psi_j(k,a) \tag{29}$$

where $e_n(k) = e_n(k\,\Delta t)$ is the kth sample of $e_n(t)$.

Using the discrete Laguerre basis functions to approximate the continuous ones, (20) becomes

$$V_{m,i}^{E} = \sum_{k=0}^{\infty} \phi_i(k) V_m^{E}(k) \tag{30}$$

The integration is replaced by the summation. This approach can reduce the computational load and can avoid errors introduced in numerical integration.

The continuous Laugerre transform of a rectangular pulse can be analytically computed as

$$c_j = \int_{t_1}^{t_2} \phi_j(st)dt = -\frac{2}{s}\phi_j(st)\Big|_{t_1}^{t_2} - 2\sum_{k=0}^{j-1} c_k \tag{31}$$

where t_1 and t_2 are the start and stop time for the pulse with an amplitude of 1V. The discrete Laguerre transform of a rectangular pulse can be computed as

$$d_j = \sum_{k=k_1}^{k_2} \psi_j(k) = \frac{a\left[\psi_j(k_1) - \psi_j(k_2)\right] - \left[\psi_{j-1}(k_1) - \psi_{j-1}(k_2)\right]}{1-a} - d_{j-1} \tag{32}$$

where k_1 and k_2 are the sampled values at the start and stop time. As a reference, we compare the coefficients of the Laguerre transforms with that from a Fourier series. It's known the Fourier series for a rectangular pulse can be written as

$$a_0 = \frac{2(t_2 - t_1)}{T}, a_j = \frac{-\sin(2k\pi t_1 / T) + \sin(2k\pi t_2 / T)}{k\pi}, b_j = \frac{\cos(2k\pi t_1 / T) - \cos(2k\pi t_2 / T)}{k\pi}. \tag{33}$$

The transform data for the three kinds for a rectangular pulse with t_1 = 1 lm (light meter), t_2 = 2 lm, time span T = 10*lm* are shown in Fig. 2. We can see that for large orders, the values of the discrete Laugerre transform approaches to zero exponentially while the other 2 are oscillating.

Reconstruction of the original signal by a discrete Laguerre transforms is more convergent than the continuous transforms, which is also shown in Fig. 3. Unlike the other 2 transforms, the reconstructed signal by the discrete transform doesn't display the well-known Gibbs phenomenon.

4. Numerical Simulation

In this paper, an example of scattering analysis of a dipole is shown as in Fig. 4. The dipole is a conducting strip with one meter long. A rectangular incident plane wave in the time domain as shown in Fig. 3 is propagating along the X axis. We analyze the induced current at the middle edge of the dipole. Figure 5 shows the comparison of the currents obtained by continuous and discrete Laugerre basis functions. When we enlarge the results in early time in Fig. 6, we can see the response obtained by discrete Laugerre functions is zero when there's no input incident wave. It's more reasonable than that obtained by the continuous basis functions. The oscillation shown in the early time for the continuous basis functions is caused by the Gibbs phenomenon.

5. Conclusions

In this paper we present the discrete Laugerre basis functions in the solution of TD-EFIE. The discontinuous rectangular incident wave is used in the scattering analysis. The discrete basis functions are superior to its continuous counterpart in the fact of representation the incident wave. It's more efficient in computing the transform data and the Gibbs phenomenon doesn't exists if we use the discrete basis functions. Since the discrete Laguerre functions cannot approximate its continuous counterpart in higher orders, more work is to be done for the improvement of the algorithm.

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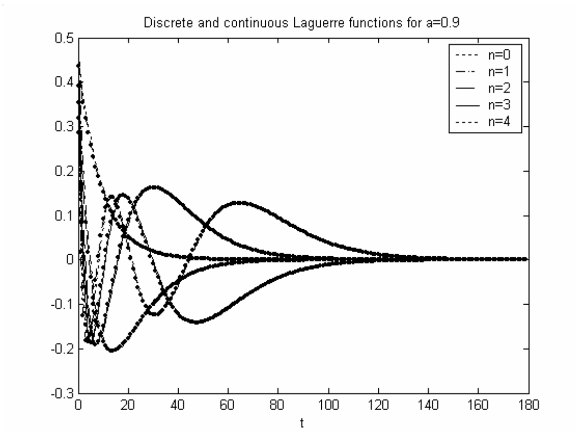

Fig. 1. Continuous and discrete basis functions.

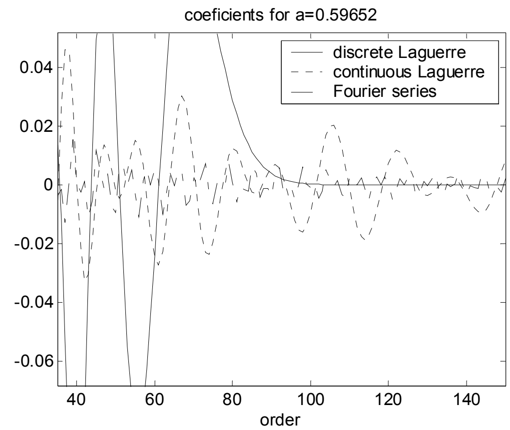

Fig. 2. Transform data for rectangular pulse

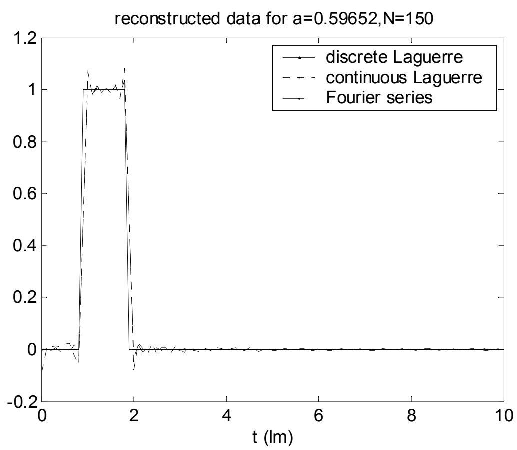

Fig. 3. Reconstructed signals

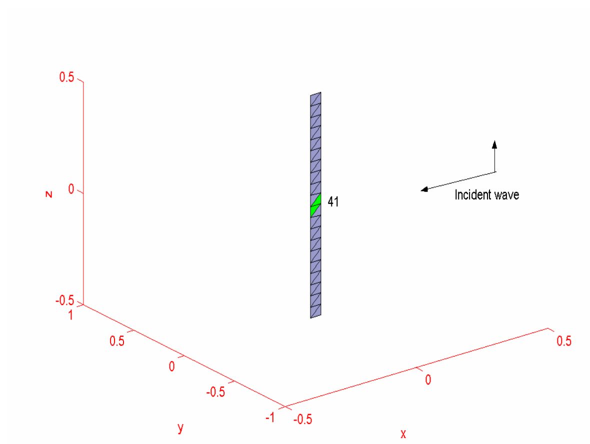

Fig. 4. The scattering structure

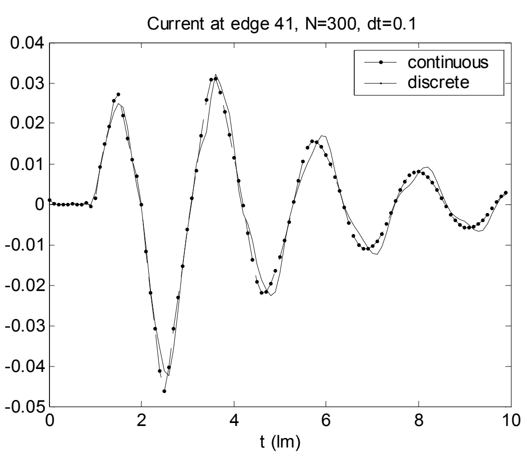

Fig. 5. Output of the time domain current.

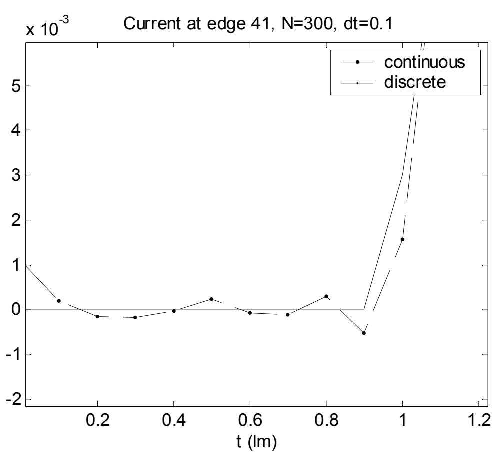

Fig. 6. Output of the time domain current (enlarged).

